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ON THE EFFECT OF WEAK NON-LINEARITIES ON LINEAR CONTROLLABILITY AND OBSERVABILITY NORMS, AN INVARIANT MANIFOLD APPROACH

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Many weakly non-linear structures exhibit normal modes that are analogous to linear modes of linear oscillatory systems. As in the testing of linear modal systems, optimal sensor and actuator placement is important for obtaining the best test results. Such optimal placement is also important in control. This paper addresses this issue by extending the concepts of modal controllability and observability norms developed for linear systems to weakly non-linear systems that exhibit non-linear normal modes.

1. INTRODUCTION

The concept that non-linear modes with non-linear modal equations exist for some set of non-linear systems has been accepted intuitively by many for quite some time. It was not until 1964 when Rosenberg presented the first paper on non-similar normal modes that it became possible to solve even the simplest non-similar normal mode system [1]. Many perturbation methods have been developed to approximate the deviation of a non-linear mode from a corresponding linear mode. Other methods [2, 3] showing the potential for mode bifurcation in strongly non-linear systems have been developed as well. Here only weakly non-linear systems in which the non-linear modes are non-linear extensions of the linear modes of the linearized system are considered. Thus, at low amplitudes, these results reduce to those of a purely linear analysis. Although algebraically tedious, this method lends itself to programming using algebraic manipulation packages such as Mathematica® [4], MAPLE®, and MACSYMA®. The following applies the invariant manifold method of Shaw and Pierre [5] to find the modal forces acting on non-linear modal equations for weakly non-linear systems to which their method is applicable.

2. AN OVERVIEW OF NON-LINEAR NORMAL MODES

Most dynamicists are, in general, comfortable with the notion of a linear normal mode. The response of a system which is moving in a linear normal mode can be written in the

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vector form

$$\mathbf{x}(t) = \mathbf{x}|_{t0}\, e^{-t/\tau} \sin(\omega t + \Phi) \tag{1}$$

where $\mathbf{x}|_{t0}$ is the mode shape or the spatial part of the modal dynamics, t is time, τ is the time constant at the mode (infinity for the undamped system), ω is the natural frequency of the mode, and Φ is a phase angle.

Unlike the linear normal mode, non-linear normal modes are not quite as readily accepted. Non-linear normal modes are defined to be either ''similar'' or ''non-similar''. A similar non-linear normal mode is one in which shape is not dependent on the modal amplitude, and thus is ''similar'' to a linear mode. In a ''non-similar'' non-linear normal mode, the mode shape is not linear. For instance, a two-degree-of-freedom system could have the mode shapes $x_2 = f_1(x_1, \dot{x}_1)$ and $\dot{x}_2 = f_2(x_1, \dot{x}_1)$.

When a system is moving in a mode it moves along a ''modal curve''. For a modally damped system, the distance traveled along the modal curve decreases with each cycle and the modal curve can change with the amplitude of oscillation. Note that for a conservative system the modal curve is dependent on the modal amplitude as well, but since the total energy remains constant, the modal amplitude remains constant, and the modal curve will remain constant for all time. A non-modally damped system moving in a single mode will not follow the same line in both directions but will instead follow a curved trajectory which spirals toward static equilibrium. This curved trajectory envelopes what would be the undamped modal curve.

The existing definition (Vakakis, [6, p. 8] and Rosenberg [1]) for a non-linear normal mode states that a discrete N-degree-of-freedom system is oscillating in a normal mode if all of the motions are periodic of the same period, all of the co-ordinates reach their extreme values at the same time, and for any given amplitude of oscillation the co-ordinate displacements can be related by a functional relationship of the form

$$x_i = f_i(u(t)), \tag{2}$$

where x_i is the displacement of the ith co-ordinate, u is the modal displacement, and f_i is the relationship between them. The definition is not general in the sense that the co-ordinates may not reach their extreme simultaneously under certain damping conditions. These conditions are still not understood well and are analogous to the modal damping conditions for linear systems [7]. Shaw and Pierre [5] demonstrate that the modal curve is dependent on the modal velocity as well as the modal displacement. In conservative systems the modal velocity contribution may be eliminated by applying conservation of energy [8]. However, equation (2) must still be dependent on some quantity other than the modal displacement and thus the existing definition is lacking. A simple example of a system oscillating in a modally damped motion is a single-degree-of-freedom mass spring system with velocity squared damping. Each cycle of the motion will exhibit a slightly different period during the decay. Also, complex modes of linear systems are not modes by this definition of non-linear normal modes since the motion of the system is not in unison. This is accounted for by Shaw and Pierre [9] who developed the capability of determining the non-linear modal equivalent to the linear complex mode. They redefined a normal mode as ''a motion which takes place on a two-dimensional invariant manifold in the system's phase space. This manifold has the following properties: it passes through a stable equilibrium point of the system and, at that point, it is tangent to a plane which is an eigenspace of the system linearized about the equilibrium.'' In order to provide consistency with linear terminology and improve the definitions of non-linear modes, the following definitions were proposed by Slater and Inman [10]:

Definition one: a system is oscillating in a *normal mode* if the motion of any point $(u_{\hat{x}}, u_{\hat{y}}, u_{\hat{z}})$ in three-dimensional space $(\hat{x}, \hat{y}, \hat{z})$ can be described by the equation

$$\begin{bmatrix} u_{\hat{x}}(\hat{x}, \hat{y}, \hat{z}) \\ u_{\hat{y}}(\hat{x}, \hat{y}, \hat{z}) \\ u_{\hat{z}}(\hat{x}, \hat{y}, \hat{z}) \end{bmatrix} = \begin{bmatrix} f_{\hat{x}}(u(t), v(t)) \\ f_{\hat{y}}(u(t), v(t)) \\ f_{\hat{z}}(u(t), v(t)) \end{bmatrix}. \tag{3a}$$

where $(\hat{x}, \hat{y}, \hat{z})$ represents the location of the point on the structure in three-dimensional space, $u_{\hat{x}}$, $u_{\hat{y}}$ and $u_{\hat{z}}$ represents the deflections in the x, y, and z directions, and $f_{\hat{x}}, f_{\hat{y}}$ and $f_{\hat{z}}$ relate the deflections to the modal co-ordinates u and $v = \dot{u}$. This represents the two-dimensional invariant manifold described by Shaw and Pierre [5].

Definition two: if the function f_i relating the displacements u_i to the modal co-ordinates are linear and the modal equations in u and v are linear then the mode is a *linear normal mode*.

Definition three: if the functions f_i relating the displacement u_i to the modal co-ordinate are linear and the modal equations in u and v are non-linear then the mode is a *similar non-linear normal mode*. This corresponds to the definition put forth by Rosenberg [1] that if the modal curves corresponding to a non-linear normal mode are straight, then the mode is called ''similar.''

Definition four: if the functions f_i relating the displacements u_i to the modal co-ordinate are non-linear then the mode is a *non-similar non-linear normal mode*. This corresponds to the definition put forth by Rosenberg [1] that if the modal curves corresponding to a non-linear normal mode are not straight, then the mode is called ''non-similar.''

Definition five: if the trajectory of a non-linear normal mode passes through static equilibrium it is an *equal phase non-linear normal mode*. For a linear normal mode this type of mode is usually called a ''real'' mode.

Definition six: if the trajectory of a non-linear normal mode does not pass through static equilibrium it is a 'non-equal phase non-linear normal mode. In a linear system this type of mode is called a ''complex'' mode.

The following sections describe the method developed by Shaw and Pierre [5] for determining non-linear normal modes. As will become evident, the method allows for the solution of non-linear normal modes which were previously undefined by the Rosenberg definition.

3. NORMAL MODES OF NON-LINEAR SYSTEMS

The method used for determining the non-linear normal modes of the system is the method of Shaw and Pierre [5]. Readers are referred to this work and others [3, 6, 8, 9, 11–17] for determining non-linear normal modes.

Using the same notation as Shaw and Pierre [5], the displacements and velocities of a system, $\mathbf{z} = [x_1, \dot{x}_1, x_2, \dot{x}_2, \ldots, x_N, \dot{x}_N]^{\mathrm{T}}$, is equal to $[x_1, y_1, x_2, y_2, \ldots, x_N, y_N]^{\mathrm{T}}$, the motion in a single non-linear normal mode, and can be written as a function of the modal displacement, u, and the modal velocity, v, as

$$\begin{bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ \vdots \\ x_N \\ y_N \end{bmatrix} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ a_{12} & a_{22} \\ b_{12} & b_{22} \\ \vdots & \vdots \\ a_{1N} & a_{2N} \\ b_{1N} & b_{2N} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ a_{32}u + a_{42}v & a_{52}v \\ b_{32}u + b_{42}v & b_{52}v \\ \vdots & \vdots \\ a_{3N}u + a_{4N}v & a_{5N}v \\ b_{3N}u + b_{4N}v & b_{5N}v \end{bmatrix} \right.$$

$$\left. + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ a_{62}u^2 + a_{82}v^2 & a_{72}u^2 + a_{92}v^2 \\ b_{62}u^2 + b_{82}v^2 & b_{72}u^2 + b_{92}v^2 \\ \vdots & \vdots \\ a_{6N}u^2 + a_{82}v^2 & a_{7N}u^2 + a_{9N}v^2 \\ b_{6N}u^2 + b_{8N}v^2 & b_{7N}u^2 + b_{9N}v^2 \end{bmatrix} \begin{matrix} \\ \\ + \cdots \\ + \cdots \\ \\ \\ \\ \end{matrix} \right\} \begin{bmatrix} u \\ v \end{bmatrix}, \qquad (4a)$$

or more compactly as

$$\mathbf{z} = \mathbf{m}\begin{bmatrix} u \\ v \end{bmatrix} = [\mathbf{m}_0 + \mathbf{m}_1(u, v) + \mathbf{m}_2(u, v) + \cdots]\begin{bmatrix} u \\ v \end{bmatrix}, \qquad (4b)$$

where u and v are the modal displacement and velocity, $\mathbf{m}$ is a $2N \times 2$ matrix and $\mathbf{m}_0$, $\mathbf{m}_1$, and $\mathbf{m}_2$ are also $2N \times 2$ matrices. The matrix $\mathbf{m}_0$ is the linear component of the non-linear modes and $\mathbf{m}_1$ and $\mathbf{m}_2$ represent the quadratic and cubic terms, respectively. This representation is not unique but is used to facilitate co-ordinate transformation later.

The matrix $\mathbf{m}_0$ represents the mode shapes common to linear systems. For a linear system, all other matrices $\mathbf{m}_j$ are zero. For the linear undamped or modally damped system, the cross terms a_{2i} and b_{1i} are zero while $a_{1i} = b_{2i}$ is the usual amplitude ratio relating the ith-degree-of-freedom to the modal amplitude. For a normally damped system, the cross terms are generally non-zero and thus represent the effect of complex modes in terms of real numbers.

Now the complete non-linear modal matrix $\mathbf{M}$ can be assembled from the modal vectors $\mathbf{m}$. The modal matrix $\mathbf{M(w)}$ is then

$$\mathbf{M(w)} = [{}_1\mathbf{m} \quad {}_2\mathbf{m} \quad {}_3\mathbf{m} \quad \ldots {}_N\mathbf{m}], \qquad (5)$$

where ${}_i\mathbf{m}$ represents the modal vector for the ith mode and $\mathbf{w} = [u_1, v_1, u_2, v_2, \ldots, u_N, v_N]^{\mathrm{T}}$, where u_i and v_i are the modal displacement and velocity, respectively, for the ith mode. The complete transformation from modal to physical co-ordinate can now be written

$$\mathbf{z} = \mathbf{M(w)w} = \tilde{\mathbf{M}}\mathbf{(w)} \qquad (6)$$

The matrix $\mathbf{M(w)}$ can be subdivided in the same manner as the vector $\mathbf{m(w)}$ was in equation (4b). This gives

$$\mathbf{M(w)} = \mathbf{M}_0 + M_1\mathbf{(w)} + \mathbf{M}_2\mathbf{(w)}. \qquad (7)$$

If one were to write out $\mathbf{M}(\mathbf{w})$ in general form, the coefficients a_{ij} and b_{ij} will in general be different for each mode. It is suggested that the notation a_{ijk} and b_{ijk} be used where the index k represents the mode number. The importance of this form will become apparent in the next section.

4. TRANSFORMATION FROM PHYSICAL TO NON-LINEAR MODAL CO-ORDINATES

It must be made clear that these transformations are an extension of linear theory but that superposition does not apply in the linear sense. The non-linear co-ordinate transformation is analogous to the well known linear transformation only in that it allows a multiple-degree-of-freedom system to be viewed as a number of simpler single-degree-of-freedom systems. The non-linear co-ordinates are curvilinear and independent, forming basis co-ordinates for the non-linear state space. They are guaranteed to be locally orthogonal only at origin where they reduce to the linear modes. The best proof of the validity of this technique is a set of Poincaré maps demonstrating the lack of any other stable non-linear normal modal curves (not stability of the non-linear modal equation). However, weakly non-linear systems (i.e., those in which the linear terms are dominant) do not exhibit more than n non-linear modal curves, n being the number of degrees-of-freedom. Vakakis [3] has shown that even a minuscule linear stiffness component in each stiffness term destabilizes the modal lines resulting from bifurcation of the normal modes.

These single-degree-of-freedom systems can be solved and their solutions combined via the non-linear co-ordinate transformations to yield the total system dynamics. For linear systems, the transformation between modal and physical co-ordinates is given by the matrix $\mathbf{M}_0$. The transformation from the physical to the modal co-ordinate for the non-linear system is not the same as the transformation from the modal to the physical co-ordinates due to the modal amplitude dependence of the transformation. For the transformation from modal to physical co-ordinates, it is assumed that the modal amplitudes are known. Since the transformation matrix is written in terms of the modal amplitudes, the transformation can be carried out (see equation (6)). In making the transformation from physical to modal co-ordinates the matrix $\mathbf{M}(\mathbf{w})$ cannot simply be inverted in order to accomplish the transformation because the modal amplitudes are not known and therefore $\mathbf{M}(\mathbf{w})$ cannot be evaluated. The transformation method developed by Shaw and Pierre [5] for systems with only cubic non-linearities is shown below (a more general method can be seen in the paper by Shaw and Pierre).

By beginning by premultiplying equation (6) by $\mathbf{M}(\mathbf{w})$ and expanding the inverse yields

$$\begin{aligned}\mathbf{w} &= \{M_0 + M_2(\mathbf{w})\}^{-1}\mathbf{z} = \{\mathrm{I} + \mathrm{M}_0^{-1}\mathrm{M}_2(\mathbf{w})\}^{-1}M_0^{-1}\mathbf{z} \\ &= \{I - M_0^{-1}M_2(\mathbf{w})\}M_0^{-1}\mathbf{z} + (I + M_0^{-1}M_2(\mathbf{w}))^{-1}(M_0^{-1}M_2(\mathbf{w}))^2M_0^{-1}\mathbf{z} \qquad (8)\end{aligned}$$

Next, the fourth order term (the second term) is dropped for the sake of simplicity. This yields

$$\mathbf{w} \approx \{I - M_0^{-1}M_2(\mathbf{w})\}M_0^{-1}\mathbf{z}. \qquad (9)$$

The right hand side still has a dependency on $\mathbf{w}$. This can be remedied first by substituting equation (9) for $\mathbf{w}$ into itself, i.e.,

$$\mathbf{w} = \{I - M_0^{-1}M_2(\{I - M_0^{-1}M_2(\mathbf{w})\}M_0^{-1}\mathbf{z})\}M_0^{-1}\mathbf{z}. \qquad (10)$$

Then, since M_2 is quadratic in its argument, the leading order argument is $M_0^{-1}\mathbf{z}$ and the dependence on $\mathbf{w}$ is pushed on to higher order terms. This results in the transformation

$$\mathbf{w} = \{I - M_0^{-1} M_2(M_0^{-1}\mathbf{z})\} M_0^{-1}\mathbf{z} = \mathbf{N}(\mathbf{z})\mathbf{z} = \tilde{\mathbf{N}}(\mathbf{z}), \tag{11}$$

which is correct up to cubic terms in $\mathbf{z}$.

The matrix $\mathbf{N}(\mathbf{z})$ now represents the transformation from physical to modal co-ordinates. Like the co-ordinate transformation matrix $\mathbf{M}(\mathbf{w})$ it can be broken into terms of different order, i.e.,

$$\mathbf{N}(\mathbf{z}) = N_0 + N_1(\mathbf{z}) + N_2(\mathbf{z}), \tag{12}$$

where N_0 represents the linear part of the transformation, $N_1(\mathbf{z})$ the quadratic part of the transformation and so on.

5. TRANSFORMATION OF NON-LINEAR STATE EQUATIONS FROM PHYSICAL TO MODAL CO-ORDINATES

Motivated by the use of the state space representation in control theory, the non-linear state space equations can be written in physical co-ordinates as

$$\dot{a} = \bar{\mathbf{A}}(\mathbf{z}, \mathbf{u}) = \tilde{\mathbf{A}}(\mathbf{z}) + \mathbf{B}(\mathbf{z})\hat{\mathbf{u}}_{\dot{z}}, \qquad \mathbf{y} = \tilde{\mathbf{C}}(\mathbf{z}, \dot{\mathbf{z}}), \tag{13, 14}$$

where $\tilde{\mathbf{A}}(\mathbf{z})$ is the state function vector and $\tilde{\mathbf{C}}(\mathbf{z}, \dot{\mathbf{z}})$ is the output function vector, and $\mathbf{u}$ represents a vector of arbitrary excitations. Taking the derivative of equation (6) with respect to time and applying the chain rule yields

$$\tilde{\mathbf{z}} = (\partial \tilde{\mathbf{M}}(\mathbf{w})/\partial \mathbf{w})\dot{\mathbf{w}}, \tag{15}$$

Substituting equations (6) and (15) into equations (13) and (14) gives

$$\dot{\mathbf{w}} = (\partial \tilde{\mathbf{M}}(\mathbf{w})/\partial \mathbf{w})^{-1}\tilde{\mathbf{A}}(\tilde{\mathbf{M}}(\mathbf{w}), \mathbf{u}) = \tilde{\mathbf{A}}_m(\mathbf{w}) + \mathbf{u}_m(\mathbf{w}, \mathbf{u}) \tag{16}$$

and

$$\mathbf{y} = \tilde{\mathbf{C}}(\tilde{\mathbf{M}}(\mathbf{w}), (\partial \tilde{\mathbf{M}}(\mathbf{w})/\partial \mathbf{w})\dot{\mathbf{w}}) = \tilde{\mathbf{C}}_m(\mathbf{w}, \dot{\mathbf{w}}), \tag{17}$$

where $\mathbf{u}_m(\mathbf{w}, \mathbf{u})$ represents the amplitude dependent modal forces, $\tilde{\mathbf{A}}_m(\mathbf{w})$ represents the modal functions, and $\tilde{\mathbf{C}}_m(\mathbf{w})$ represents the observation function in modal co-ordinates.

6. TRANSFORMATION OF NON-LINEAR STATE EQUATIONS FROM MODAL TO PHYSICAL CO-ORDINATES

In transforming the equations from modal co-ordinates to physical co-ordinates, the same procedure is used as in the previous section. The equations of motion in modal co-ordinates are given by

$$\dot{\mathbf{w}} = \tilde{\mathbf{A}}_m(\mathbf{w}) + \mathbf{u}_m(\mathbf{w}), \qquad \mathbf{y} = \tilde{\mathbf{C}}_m(\mathbf{w}, \dot{\mathbf{w}}), \tag{18, 19}$$

where $\mathbf{u}_m(\mathbf{w})$ represents a vector of arbitrary modal forces. From equation (11),

$$\mathbf{w} = \tilde{\mathbf{N}}(\mathbf{z}), \tag{20}$$

Taking the time derivative and applying the chain rule to equation (20) gives

$$\dot{\mathbf{w}} = (\partial \tilde{\mathbf{N}}(\mathbf{z})/\partial \mathbf{z})\dot{\mathbf{z}}, \tag{21}$$

where $\partial\tilde{\mathbf{N}}(\mathbf{z})/\partial\mathbf{z}$ is the Jacobian of $\tilde{\mathbf{N}}(\mathbf{z})$ with respect to $\mathbf{z}$. Substituting equations (11) and (21) into the modal equations of motion and premultiplying by $(\partial\tilde{\mathbf{N}}(\mathbf{z})/\partial\mathbf{z})^{-1}$ yields the state equations in physical co-ordinates as

$$\dot{\mathbf{z}} = (\partial\tilde{\mathbf{N}}(\mathbf{z})/\partial\mathbf{z})^{-1}\tilde{\mathbf{A}}_m(\tilde{\mathbf{N}}(\mathbf{z})) + (\partial\tilde{\mathbf{N}}(\mathbf{z})/\partial\mathbf{z})^{-1}\mathbf{u}_m(\tilde{\mathbf{N}}(\mathbf{z})) = \tilde{\mathbf{A}}'(\mathbf{z}, \mathbf{u}) \tag{22}$$

and

$$\mathbf{y} = \tilde{\mathbf{C}}_m(\tilde{\mathbf{N}}(\mathbf{z}), (\partial\tilde{\mathbf{N}}(\mathbf{z})/\partial\mathbf{z})\dot{\mathbf{z}}) = \tilde{\mathbf{C}}'(\mathbf{z}, \dot{\mathbf{z}}). \tag{23}$$

Here the prime symbol represents the possibility that the transformation does not allow the true equations of motion to be recovered in transforming them from modal to physical co-ordinates. For example, if a power series approximation for the modes is used, as suggested by Shaw and Pierre, then for practical purposes the series must be truncated for most problems. Since the modes are an approximation of the true modes, only an approximation of the true modal equations will be found. Here the "true" modal equations and the approximation of them have not been distinguished since in this work they will always be derived from the equations of motion using a power series and thus will in general be approximate, the exception being similar normal modal systems. Since the equations of motion are derived in physical space and they can also be found again by the transformation from modal co-ordinates to physical co-ordinates it is important to point out that the physical space equations of motion derived from the modal equations will not be as correct as when derived from first principles. However, the backwards transformation (from modal to physical co-ordinates) can thus be useful in determining how accurate the transformations are.

7. MODAL CONTROLLABILITY OF A NON-LINEAR NORMAL MODAL SYSTEM

Considering linear controllability, the *ideal controllability* case is that in which each modal equation can be excited individually. If the force vector $\hat{\mathbf{u}}_z$ is written

$$\hat{\mathbf{u}}_z = P_{um}(\mathbf{w})\hat{\mathbf{u}}_{m_{even}}, \tag{24}$$

where $\hat{\mathbf{u}}_{m_{even}}$ is an $N \times 1$ vector of independent modal forces and $P_{um}(\mathbf{w})$ is an $N \times N$ matrix defined as

$$P_{um}(\mathbf{w}) = \mathbf{B}(\tilde{\mathbf{M}}(\mathbf{w}))^{-1}_{even}(\partial\tilde{\mathbf{M}}(\mathbf{w})/\partial\mathbf{w})_{even}, \tag{25}$$

then the non-linear normal modes are individually controllable. Only the even rows and columns of the matrices are considered because the odd numbered rows and columns represent identity statements which are a result of writing the second order equations of motion in first order form. In order for equation (25) to have a solution, the matrix $\mathbf{B}(\tilde{\mathbf{M}}(\mathbf{w}))_{even}$ must be non-singular. Since this criteria will seldom be met, a more useful way to look at the controllability of a system is to look at controllability norms similar to those defined by Hughes and Skelton [18], Hamdan and Nayfeh [19], and Takahashi *et al*. [20] for linear systems.

Just as in linear systems, it is difficult to obtain a useful meaning from a controllability norm if the inputs are not normalized. Thus a matrix $\mathbf{P}_n$ is defined for the non-linear system identical to the linear system such that

$$\mathbf{P}_{n_{i,i}} = 1/\max(\hat{u}_i) \tag{26}$$

and $\mathbf{B}_{mn}(\mathbf{w})$ and $\hat{\mathbf{u}}_n$ defined such that

$$(\partial\tilde{\mathbf{M}}(\mathbf{w})/\partial\mathbf{w})^{-1}\mathbf{B}(\tilde{\mathbf{M}}(\mathbf{w}))\hat{\mathbf{u}}_z - [(\partial\tilde{\mathbf{M}}(\mathbf{w})/\partial\mathbf{w})^{-1}\mathbf{B}(\tilde{\mathbf{M}}(\mathbf{w}))\mathbf{P}_n^{-1}][\mathbf{P}_n\hat{\mathbf{u}}_z] = [\mathbf{B}_{mn}(\mathbf{w})][\hat{\mathbf{u}}_{zn}], \quad (27)$$

where $\mathbf{u}_n$ represents the normalized force vector and $\mathbf{B}_{mn}(\mathbf{w})$ represents the control input matrix. The element $\mathbf{B}_{mn_{q,r}}(\mathbf{w})$ represent the scaling of the rth input force to the qth mode. Note that $\mathbf{B}_{mnq,r}(\mathbf{w})$ represents the element of $\mathbf{B}_{mn}(\mathbf{w})$ in the $2q$th row and rth column. Thus

$$\mathbb{C}_{q,r} = |\mathbf{B}_{mn_{q,r}}(\mathbf{w})| \quad (28)$$

is defined to be the controllability norm of the qth mode from the rth input. This norm also represents the maximum force which may be applied to the rth mode from the qth input. It is also possible to define the controllability norm of the qth mode from set R of inputs. This is given by

$$\mathbb{C}_{q,R} = (\mathbf{B}_{mn_{q,r}}(\mathbf{w})B^{\mathrm{T}}_{mn_{q,R}}(\mathbf{w}))^{1/2}. \quad (29)$$

For the case of repeated modes, the matrix $\mathbf{B}_{mn_{q,R}}(\mathbf{w})$ can be defined to be the Rth columns and the $2q$th rows (corresponding to the qth mode), of the matrix $\mathbf{B}_{mn}(\mathbf{w})$. The modal controllability norm is then given by

$$\mathbb{C}_{q,R} = \det(\mathbf{B}_{mn_{q,R}}(\mathbf{w})\mathbf{B}^{\mathrm{T}}_{mn_{q,R}}(\mathbf{w}))^{1/2Nq}, \quad (30)$$

where N_q is the multiplicity of the qth mode. Note that this definition breaks down in the absence of pure modal motion and thus becomes an approximation since the modal equations are not completely decoupled. For a linear system, these results collapse to the results of Hughes and Skelton [18].

8. MODAL OBSERVABILITY OF A NON-LINEAR NORMAL SYSTEMS

A system is said to be ideally observable if from the outputs of the system, all of the states of the system can be determined. Thus the inverse function

$$\mathbf{w} = \tilde{\mathbf{Q}}(\hat{\mathbf{y}}, \hat{\mathbf{u}}_z) \quad (31)$$

is defined such that

$$\hat{\mathbf{y}} = \tilde{\mathbf{C}}_m(\tilde{\mathbf{Q}}(\hat{\mathbf{y}}, \hat{\mathbf{u}}_z), \dot{\tilde{\mathbf{Q}}}(\hat{\mathbf{y}}, \hat{\mathbf{u}}_z)). \quad (32)$$

If a function $\tilde{\mathbf{Q}}(\hat{\mathbf{y}}, \hat{\mathbf{u}}_z)$ can be found which satisfies equation (32) then the system is said to have "ideal observability." Since this will not be possible in the majority of systems it is useful to quantify how observable each mode is from each sensor. In Slater [21] the modal controllability for a linear system was defined based on the sensitivity of $\mathbf{y}$ to the modal vector $\mathbf{w}$ after scaling the output equation with respect to the sensitivities of the sensors. The same arguments for scaling the linear output equation hold for the non-linear output equation. Thus, the output equation (14) becomes

$$\hat{\mathbf{y}}_n = \tilde{\mathbf{C}}_{mn}(\mathbf{w}, \hat{\mathbf{u}}_z), \quad (33)$$

where $\hat{\mathbf{y}}_n = \mathbf{P}_{ncn}^{-1}\hat{\mathbf{y}}$ and $\tilde{\mathbf{C}}_{mn}(\mathbf{w}, \hat{\mathbf{u}}_z) = \mathbf{P}_{ncn}^{-1}\tilde{\mathbf{C}}_n(\mathbf{w}, \hat{\mathbf{u}}_z)$ and $\mathbf{P}_{ncn}^{-1}$ is a diagonal matrix consisting of the sensitivities of the sensors. The sensitivities of $\hat{\mathbf{y}}_n$ to $\mathbf{w}$ is the Jacobian of $\hat{\mathbf{y}}_n$ with respect to $\mathbf{w}$:

$$S(\mathbf{w}, \hat{\mathbf{u}}_z) = \partial\hat{\mathbf{y}}_n/\partial\mathbf{w} = \partial\tilde{\mathbf{C}}_{mn}(\mathbf{w}, \hat{\mathbf{u}}_z)/\partial\mathbf{w}. \quad (34)$$

The observability of the ith-degree-of-freedom the rth sensor may then be defined as

$$\mathbf{Ø}_{r,i} = |S_{r,i}(\mathbf{w}, \hat{\mathbf{u}}_z)|, \tag{35}$$

as in the linear system. This definition, however, neglects the detrimental effects of a sensor from which the readings are an even function of a mode. For example, an accelerometer sensing radial acceleration of a rotating shaft can sense rotation but cannot determine the direction of the rotation. Depending on the purpose of the output, a more useful norm might be

$$\mathbf{Ø}_{odd_{r,i}} = |\mathcal{S}_{r,i}(\mathbf{w}, \hat{\mathbf{u}}_z)|, \tag{36}$$

where

$$\mathcal{S}(\mathbf{w}, \hat{\mathbf{u}}_z) = \tfrac{1}{2}(S(\mathbf{w}, \hat{\mathbf{u}}_z) + S(-\mathbf{w}, \hat{\mathbf{u}}_z)). \tag{37}$$

Defining the norm in this fashion causes only the odd part of the output to be considered in the observability norm. A gross measure of observability from the sensors R can then be defined as

$$\mathbf{Ø}_{R,i} = (\mathcal{S}_{R,i}(\mathbf{w}, \hat{\mathbf{u}}_z)^{\mathrm{T}}\mathcal{S}_{R,i}(\mathbf{w}, \hat{\mathbf{u}}_z))^{1/2}, \tag{38}$$

where $\mathcal{S}_{R,i}(\mathbf{w}, \hat{\mathbf{u}}_z)$ is a vector containing the Rth element of the ith column of $\mathcal{S}(\mathbf{w}, \hat{\mathbf{u}}_z)$.

9. EXAMPLE

Consider the two-degree-of-freedom system of Figure 1 where only the second degree of freedom can be actuated. The equations of motion for the system are then

$$\begin{bmatrix} \dot{x}_1 \\ \dot{y}_1 \\ \dot{x}_2 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ -2x_1 - 0{\cdot}5x_1^3 + x_2 \\ y_2 \\ x_1 - 2x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \hat{u}_z$$

and

$$\hat{\mathbf{y}} = \mathbf{Iz}.$$

Using the method of Shaw and Pierre [5], the equations of motion in modal

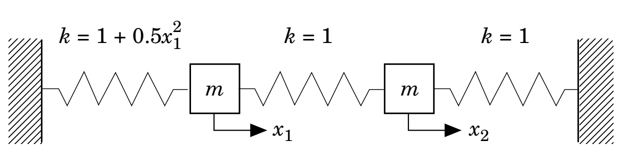

Figure 1. A two-degree-of-freedom oscillator with a cubic stiffness.

co-ordinates are

$$\begin{bmatrix} \dot{u}_1 \\ \dot{v}_1 \\ \dot{u}_2 \\ \dot{v}_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ -u_1 - \frac{1}{3}u_1^3 + \frac{1}{4}u_1 v_1^2 - \frac{3}{4}u_1^2 u_2 - \frac{3}{4}u_1 u_2^2 \\ v_2 \\ -3u_2 - \frac{4}{13}u_2^3 + \frac{3}{52}u_2 v_2^2 - \frac{3}{4}u_1^2 u_2 - \frac{3}{4}u_1 u_2^2 \end{bmatrix} + \begin{bmatrix} \hat{u}_{m1} \\ \hat{u}_{m2} \\ \hat{u}_{m3} \\ \hat{u}_{m4} \end{bmatrix},$$

where the modal control forces are

$$\hat{u}_{m1} = -\hat{u}_{m3} = (0{\cdot}125 u_1 v_1 + 0{\cdot}028 u_2 v_2)\hat{u}_z$$

and

$$\hat{u}_{m2} = -\hat{u}_{m4} = (-0{\cdot}500 + 0{\cdot}0577 u_2^2 - 0{\cdot}188 v_1^2 + 0{\cdot}0433 v_2^2)\hat{u}_z.$$

The output equation is given by

$$\hat{\mathbf{y}} = \tilde{\mathbf{M}}(\mathbf{w}) = \begin{bmatrix} u_1 + u_2 \\ v_1 + v_2 \\ u_1 - u_2 + 0{\cdot}1667 u_1^3 + 1{\cdot}923 u_2^3 + 0{\cdot}25 u_1 v_1^2 + 0{\cdot}5769 u_2 v_2^2 \\ v_1 - v_2 + 0{\cdot}25 v_1^3 + 0{\cdot}05769 v_2^3 + 0{\cdot}2308 u_2^2 v_2 \end{bmatrix},$$

mode 1 is given by

$$\begin{bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \end{bmatrix} = \tilde{\mathbf{M}}(\mathbf{w})\big|_{u_2=0, v_2=0} = \begin{bmatrix} u_1 \\ v_1 \\ u_1 + \frac{1}{6}u_1^3 + 0{\cdot}25 u_1 v_1^2 \\ v_1 + 0{\cdot}25 v_1^3 \end{bmatrix},$$

and mode 2 is given by

$$\begin{bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \end{bmatrix} = \tilde{\mathbf{M}}(\mathbf{w})\big|_{u_1=0, v_1=0} = \begin{bmatrix} u_2 \\ v_2 \\ -u_2 + 0{\cdot}1923 u_2^3 + 0{\cdot}5769 u_2 v_2^2 \\ -v_2 + 0{\cdot}05769 v_2^3 + 0{\cdot}2308 u_2^2 v_2 \end{bmatrix}.$$

From equation (28) the matrix of amplitude dependent controllability norms is given as

$$\mathcal{C} = \text{abs}\left(\begin{bmatrix} 0 & 0 \\ 0 & (0{\cdot}500 + 0{\cdot}0577 u_2^2 - 0{\cdot}188 v_1^2 + 0{\cdot}0433 v_2^2) \\ 0 & 0 \\ 0 & -(0{\cdot}500 + 0{\cdot}0577 u_2^2 - 0{\cdot}188 v_1^2 + 0{\cdot}0433 v_2^2) \end{bmatrix}\right)$$

and from equation (35) the matrix of amplitude dependent observability norms is

$$\varnothing = \text{abs}\left\{\begin{bmatrix} 1 & 0 & 1 + 0{\cdot}5u_1^2 + 0{\cdot}25v_1^2 & 0 \\ 1 & 0 & -1 + 5{\cdot}7695u_2^2 + 0{\cdot}05769v_2^2 & 0{\cdot}4616u_2v_2 \\ 0 & 1 & 0{\cdot}5u_1v_1 & 1 + 0{\cdot}75v_1^2 \\ 0 & 1 & 0{\cdot}1154u_2v_2 & -1 + 0{\cdot}2308u_2^2 + 0{\cdot}1731v_2^2 \end{bmatrix}^{\mathrm{T}}\right\}.$$

From these results, three observations can be made. From the controllability norms it can be seen that any force applied to the structure will excite each mode with equal magnitude. This result is identical to that for the linearized system, and thus the asymmetry of the non-linearity does not effect how much of a given force will be applied to each mode. From the observability norms it can be seen that for mode one a greater observability norm occurs at position 2 for large oscillations than for small oscillations and that the converse is true for the second mode. In fact, mode shape plots show that for high amplitude oscillations, the modal motions tend towards becoming localized, thus changing the ability to sense one mode as compared to the other as a function of amplitude.

10. CONCLUSION

The method of non-linear normal modes proposed by Shaw and Pierre has been extended to the forced response case and now includes the output equation as well as the state equation. The appropriate transformation have been derived and applied to a two-degree-of-freedom non-linear oscillator, illustrating that observability and controllability conditions can be established to assist in an eventual measurement and/or control procedure for non-linear modes. In addition, several new definitions have been proposed to allow non-linear mode theory to agree more precisely with the non-conservative linearized case.

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NOMENCLATURE

$_{q,r}$	element of the specified matrix corresponding to the qth row and the rth column
$_{q,R}$	element of the specified matrix corresponding to the qth row and the Rth column
$_{even}$	even numbered rows of a non-square matrix or even numbered rows and columns of a square matrix
$\tilde{\mathbf{A}}(\mathbf{z})$	state functions
$\tilde{A}_i(\mathbf{z})$	ith element of $\tilde{\mathbf{A}}(\mathbf{z})$
$A(\mathbf{z})$	$\tilde{\mathbf{A}}(\mathbf{z}) = A(\mathbf{z})\mathbf{z}$
$\tilde{\mathbf{A}}_m(\mathbf{w})$	modal state functions
$\tilde{A}_{m_i}(\mathbf{w})$	ith element of $\tilde{\mathbf{A}}_m(\mathbf{w})$
$A_m(\mathbf{w})$	$\tilde{\mathbf{A}}_m(\mathbf{w}) = A_m(\mathbf{w})\mathbf{w}$
$A_{md}(\mathbf{w})$	desired modal state matrix
$\tilde{\mathbf{A}}_{md}(\mathbf{w})$	desired modal state functions
α, β	constants representing the linear relationship between x_i and y_i
$\mathbf{B}$	usual state space $2N \times N$ matrices
$\mathbf{B}_f$	$N \times N$ force matrix for the system in linear second order form
$\mathbf{B}_{fn}$	normalized $\mathbf{B}_f$
$\mathbf{B}_{mn}$	$\mathbf{B}_m$ normalized
$\mathbf{B}_{mpn}$	$\mathbf{B}_{mp}$ normalized
$\mathbf{B}_{mp}, \mathbf{B}_m$	modal forcing matrix
$\mathbf{B}_{mp_q}$	the qth row of the matrix $\mathbf{B}_{mp}$
$\mathbf{C}_p$	output matrix for states
$\mathbf{C}_{pm}$	modal output matrix for states
$\mathbf{C}_{pmn}$	normalized modal output matrix for states
$\mathbf{C}_v$	output matrix for the derivatives of the states
$\tilde{\mathbf{C}}(\mathbf{z}, \hat{\mathbf{u}}_z)$	observation function in the physical co-ordinates
$\tilde{\mathbf{C}}_m(\mathbf{z}, \hat{\mathbf{u}}_z)$	observation function in modal co-ordinates
$\mathcal{C}_{\mathbf{q},\mathbf{r}}$	controllability norm of the qth mode from the rth actuator
$f(t)$	arbitrary forcing function
Φ	phase angle
$\mathbf{G}_z(\mathbf{z})$	modal control law function matrix
$\mathbf{G}_m(\mathbf{w})$	modal control law
$\mathbf{G}_p$	control feedback matrix for state vector
$\mathbf{G}_v$	control feedback matrix for derivative of state vector
$\mathbf{I}$	identity matrix

$\mathbf{I}_q$	identity matrix size N_q
$\mathbf{m}$	non-linear mode vector
${}_i\mathbf{m}$	non-linear mode vector of ith mode
$\mathbf{M}(\mathbf{w})$	$\mathbf{z} = \mathbf{M}(\mathbf{w})\mathbf{w}$
$\tilde{\mathbf{M}}(\mathbf{w})$	$\mathbf{z} = \tilde{\mathbf{M}}(\mathbf{w})$
$\mathbf{M}, \mathbf{D}, \mathbf{K}$	mass, damping and stiffness matrices
$N(\mathbf{z})$	$\mathbf{w} = N(\mathbf{z})\mathbf{z}$
$\tilde{\mathbf{N}}(\mathbf{z})$	$\mathbf{w} = \tilde{\mathbf{N}}(\mathbf{z})$
N_Q	number of distinct natural frequencies
$ø_{q,r}$	observability norm of the qth mode from the rth sensor
$\mathbf{P}_{fp}$	diagonal matrix of sensor sensitivities
$\mathbf{P}_n$	normalization matrix
$S(\mathbf{w}, \hat{\mathbf{u}}_z)$	sensitivity of $\mathbf{y}$ to $\mathbf{w}$
u, v	modal co-ordinates
u_i, v_i	ith modal displacement aid velocity
$\hat{\mathbf{u}}$	force vector in physical co-ordinates
$\hat{u}_i$	ith element of $\hat{\mathbf{u}}$
$u_{\hat{x}}, u_{\hat{y}}, u_{\hat{z}}$	deflections in the $\hat{x}$, $\hat{y}$, $\hat{z}$ directions
$\hat{\mathbf{u}}_m$	force vector in modal co-ordinates
$\hat{\mathbf{u}}_z$	force vector in linear second order form
$\mathbf{w}$	modal co-ordinate space
x_i, y_i	displacement and velocity of the ith degree of freedom, respectively
x_c, y_c	chosen displacement and velocity pair
$\hat{x}, \hat{y}, \hat{z}$	location of a point on a structure in three-dimensional space
X_i, Y_i	functions which relate the displacements x_i, y_i to the modal coordinates u and v
V_i	potential energy of the ith spring
$\hat{\mathbf{y}}$	output variable
$\hat{\mathbf{y}}_n$	normalized output variable
$\mathbf{z}$	physical co-ordinate space